LONG CYLINDRICAL WAVES IN A VISCOUS FLUID

(DLINNYE TSILINDRICHESKIE VOLNY V VIAZKOI ZHIDKOSTI)

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In the study of long waves on the surface of a viscous fluid it has been assumed that their oscillation frequency and velocity can be determined by the formulas for an ideal fluid. The attenuation factor can thus be determined from energy considerations.

Here we consider the equations of motion for long waves in a system of cylindrical coordinates. Formulas have been found for the oscillation frequency and attenuation factor. The free surface form, the trajectories of the fluid particles, the pressure and the velocity components were determined.

The problem of cylindrical waves on the surface of a viscous fluid of finite depth has been reduced in [1] to the solution of Equations

$$\frac{\partial v_r}{\partial t} = -\frac{1}{\rho}\frac{\partial p_1}{\partial r} + \nu\left(\Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}\right), \qquad \frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p_1}{\partial z} + \nu\Delta v_z^{(1)}$$

$$\frac{\partial v_\varphi}{\partial t} = -\frac{1}{\rho r}\frac{\partial p_1}{\partial \varphi} + \nu\left(\Delta v_\varphi - \frac{v_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}\right), \qquad \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z^1}{\partial z} = 0 \qquad (1)$$

$$\left(p_1 = p + \rho g z, \qquad \Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right)$$

with the boundary conditions

$$\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} = 0, \quad \frac{1}{r}\frac{\partial v_z}{\partial \varphi} + \frac{\partial v_\varphi}{\partial z} = 0, \quad -\frac{\partial p_1}{\partial t} + \rho g v_z + 2\mu\frac{\partial^2 v_z}{\partial z\,\partial t} = 0 \qquad \text{for } z = 0 \quad (2)$$

$$v_r = 0, \quad v_\varphi = 0, \quad v_z = 0 \qquad \text{for } z = -h \qquad (3)$$

These equations and boundary conditions can be considerably simplified if the usual assumption for long waves [2] is made.

By neglecting the inertia and viscous forces in the third equation of system (1), it can be integrated to yield

$$p_1 = p_0 + \rho g \zeta \qquad (4)$$

where ζ is the ordinate $z = \zeta(r,\varphi,t)$ of the free surface.

By eliminating the pressure from the first two equations of system (1) with the aid of (4), we introduce the new unknown $\zeta = \zeta(r,\varphi,t)$. Since $\partial\zeta/\partial t = v_z(r,\varphi,0,t)$, then multipying the last equation of (1) by dz and integrating it from $z = -h$ to $z = 0$, we obtain a third equation that contains the same unknowns ζ, v_r and v_φ

$$\frac{\partial \zeta}{\partial t} = -\frac{h}{r}\left[\frac{\partial}{\partial r}(r v_r^*) + \frac{\partial v_\varphi^*}{\partial \varphi}\right] \qquad \left(v_r^* = \frac{1}{h}\int_{-h}^{0} v_r\,dz,\ v_\varphi^* = \frac{1}{h}\int_{-h}^{0} v_\varphi\,dz\right) \tag{5}$$

This last equation enables one to eliminate ζ from the first two. Moreover, in view of the assumptions already made, we can neglect the second term in the first two boundary conditions (2).

Thus we are led to the problem of solving Equations

$$\frac{\partial^2 v_r}{\partial t^2} = gh\frac{\partial}{\partial r}\frac{1}{r}\left[\frac{\partial}{\partial r}(r v_r^*) + \frac{\partial v_\varphi^*}{\partial \varphi}\right] + \nu\frac{\partial}{\partial t}\left(\Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}\right)$$
$$\frac{\partial^2 v_\varphi}{\partial t^2} = g\frac{h}{r^2}\frac{\partial}{\partial \varphi}\left[\frac{\partial}{\partial r}(r v_r^*) + \frac{\partial v_\varphi^*}{\partial \varphi}\right] + \nu\frac{\partial}{\partial t}\left(\Delta v_\varphi - \frac{v_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}\right) \tag{6}$$

with the boundary conditions

$$\frac{\partial v_r}{\partial z} = 0, \quad \frac{\partial v_\varphi}{\partial z} = 0, \quad \text{for } z = 0, \quad v_r = 0, \quad v_\varphi = 0 \quad \text{for } z = -h \tag{7}$$

In employing the Fourier method, we will seek the solution of system (6) in the form

$$v_r = e^{kt}\cos n\varphi\,[J_{n-1}(ar) - J_{n+1}(ar)]\,Z_1(z), \quad v_\varphi = \frac{n}{r}e^{kt}\sin n\varphi J_n(ar)\,Z_2(z) \tag{8}$$

where a and n are given positive numbers, with n an integer.

Substituting this solution into (6) and using the recurrence formulas for Bessel functions, it is found that the functions Z_1 and Z_2 must satisfy Equations

$$Z_1'' - b^2 Z_1 = g\frac{a^2 h}{k\nu}Z_1^*, \quad Z_2'' - b^2 Z_2 = -g\frac{2ah}{k\nu}Z_1^* \quad \left(b^2 = a^2 + \frac{k}{\nu}, \quad Z_1^* = Z_1(z^0)\right) \tag{9}$$

The solution of system (9) can be taken in the form

$$Z_1 = Ae^{bz} + Be^{-bz} - g\frac{a^2 h}{k\nu b^2}Z_1^*, \qquad Z_2 = Ce^{bz} + De^{-bz} + g\frac{2ah}{k\nu b^2}Z_1^* \tag{10}$$

After using the boundary conditions (7) in order to determine the arbitrary constants A, B, C and D, we obtain

$$A = B = g\frac{a^2 h Z_1^*}{2k\nu b^2\cosh bh}, \qquad C = D = -g\frac{ah\,Z_1^*}{k\nu b^2\cosh bh}$$

Then, substitution of Expressions (10) for Z_1 and Z_2 into (8) leads to

$$v_r = g\frac{ha^2 Z_1^*}{k\nu b^2}e^{kt}\cos n\varphi\,[J_{n-1}(ar) - J_{n+1}(ar)]\left(\frac{\cosh bz}{\cosh bh} - 1\right)$$
$$v_\varphi = g\frac{2hanZ_1^*}{rk\nu b^2}e^{kt}\sin n\varphi J_n(ar)\left(1 - \frac{\cosh bz}{\cosh bh}\right) \tag{11}$$

The vertical component of the velocity can be obtained from the last equation in (1) by replacing v_r and v_φ in it by their values in (11) and then integrating the resulting expression with respect to z. These calculations yield

$$v_z = g\frac{2ha^3 Z_1^*}{k\nu b^2}e^{kt}\cos n\varphi J_n(ar)\left(\frac{\sinh bz + \sinh bh}{b\cosh bh} - z - h\right) \tag{12}$$

The form of the free surface can be determined by integrating Expression (12) with respect to time after having set $z = 0$. Carrying out this calculation and taking the real part, we obtain

$$\zeta(r, \varphi, t) = Ne^{\vartheta t}J_n(ar)\cos n\varphi\sin(\sigma t + \varepsilon)$$
$$\left(N = \frac{2ha\,|Z_1^*|}{\sqrt{\vartheta^2 + \sigma^2}}, \ \tan\varepsilon = \frac{\vartheta\,\mathrm{Re}\,Z_1^* + \sigma\,\mathrm{Im}\,Z_1^*}{\sigma\,\mathrm{Re}\,Z_1^* - \vartheta\,\mathrm{Im}\,Z_1^*}, \ \vartheta + i\sigma = k\right) \tag{13}$$

By substitution of this value for ζ into (4) we find the pressure

$$p_1 = p_0 + \rho g N e^{\vartheta t} J_n(ar) \cos n\varphi \sin(\sigma t + \varepsilon)$$

Considering the free surface at a sufficiently large distance from the origin of coordinates we can take only the first term in the asymptotic expansion for J_n in (13). Thus we obtain the approximate expression

$$\zeta(r, \varphi, t) \approx N\sqrt{2/\pi a r}\, e^{\vartheta t} \cos n\varphi \cos(ar - {}^1/_4\pi - {}^1/_2 n\pi) \sin(\sigma t + \varepsilon) \qquad (14)$$

It is clear from (14) that as r increases the amplitude of oscillation decreases as $1/r^{1/2}$. The nodal lines of the free surface will consist of radii with angular spacing π/n, and concentric circles with centers at the origin of coordinates and spaced at distances of π/a from each other.

It is easy to find the trajectories of the fluid particles if one first intergates Expressions (11) and (12). Because the particles perform small oscillations, the same method of integration as that in [2] leads, after some calculation, to the results

$$r - r_0 = \mathrm{Re}^{\vartheta t} \sin(\sigma t + \varepsilon_1) + R_1, \qquad \varphi - \varphi_0 = n\Phi e^{\vartheta t} \sin(\sigma t + \varepsilon_2) + n\Phi_1$$

$$z - z_0 = K e^{\vartheta t} \sin(\sigma t + \varepsilon_3) + K_1 \qquad (15)$$

where $R, R_1, \Phi, \Phi_1, K, K_1, \varepsilon_1, \varepsilon_2, \varepsilon_3$ are certain constants and r_0, φ_0 and z_0 are the coordinates of the particle in the equilibrium state. From the obtained formulas it is clear that each particle executes a damped oscillation about its equilibrium position.

The solution of the axisymmetric problem is obtained as a particular case by setting $n = 0$ in all above formulas.

The numbers $Z_1^* = Z_1(z^\circ)$ and k appearing in Formulas (11) and (12) have not yet been determined. For the determination of z°, set $z = z^\circ$ in the first of Formulas (10). This leads to the relation

$$\cosh bz^\circ = \cosh bh\left(\frac{k\nu b^2}{gha^2} + 1\right) \qquad (16)$$

As is clear from Formula (8), coefficient k indicates how the overall motion changes with time. This coefficient can be determined by means of the identity

$$v_r(r, \varphi, z^\circ, t) = \frac{1}{h}\int_{-h}^{0} v_r(r, \varphi, z, t)\, dz \qquad (17)$$

Replacing v_r in (17) by its value (11) and taking account of relation (16), an integration leads to Equation

$$ga^2(\tanh bh - bh) = k\nu b^3 \qquad (18)$$

which serves for the determination of $k = \vartheta + i\sigma$.

This equation can be simplified somewhat if instead of the unknown k we take

$$\beta = bh = h\sqrt{a^2 + k/\nu} \qquad (19)$$

By making this substitution, (18) can be written in the form

$$\beta^3(\beta^2 - \alpha^2)\nu^2 + \alpha^2 gh^3(\beta - \tanh\beta) = 0 \qquad (20)$$

Equation (20) is much simpler than Equation (2.4) obtained in [1] for the determination of the same unknown. The roots of this equation depend on two parameters. The parameter $\alpha = ah = 2\pi h/\lambda$ characterizes the ratio of the depth of fluid to the wavelength and is very small for long waves. The parameter $gh^3\nu^{-2}$ characterizes the ratio of the depth of fluid to the viscosity and can assume a wide range of values.

If $\beta = \gamma + i\delta$ is a root of Equation (20), then it follows from Formula (19) that the frequency of oscillation σ and the attenuation factor ϑ are given by Expressions

$$\sigma = 2\nu\gamma\delta h^{-2}, \qquad \vartheta = \nu\,(\gamma^2 - \alpha^2 - \delta^2)h^{-2} \tag{21}$$

from which it is evident that an oscillatory motion of the liquid will take place only when the roots of Equation (20) are complex. An analysis of the numerical solution of this equation for various values of the parameters leads to the conclusion that the roots of (20) are complex when $h^3(\nu\lambda)^{-2} \geqslant 4.84\cdot 10^{-5}$ sec^2/cm . Whence it follows that the occurrance of waves of a given length in a fluid of a higher degree of viscosity requires a large depth. By rearranging the preceding condition in the form $\lambda \leqslant 143\, h^{3/2}\nu^{-1}$, one comes to the conclusion that in a fluid of given depth the wavelength is bounded from above, whereas the waves in an ideal fluid are not subjected to any such restriction.

By examining the table of solutions of Equation (20), one also comes to the conclusion that when $\alpha^2 g h^3 \nu^{-2} \geqslant 3000$ the roots of this equation have a real part $\gamma \geqslant 5$. Since $\tanh x = 1$ when $x \geqslant 5$ (accurate to the fifth significant digit), the transcendental equation (20) can, in this case, be transformed into the algebraic equation

$$\beta^3(\beta^2 - \alpha^2)\,\nu^2 + \alpha^2 g h^3(\beta - 1) = 0 \tag{22}$$

Taking the fluid to be ideal as a first approximation, we will seek the solution of (22) in the form of the series

$$\beta = \beta_1 M + \beta_2 + \beta_3 M^{-1} + \ldots \qquad (M = \alpha^2 g h^3 \nu^{-2}) \tag{23}$$

Substitution of solution (23) into Equation (22) and identification of coefficients of like powers of M furnishes the following system of equations

$$\beta_1^5 + \beta_1 = 0, \qquad 5\beta_1^4\beta_2 + \beta_2 - 1 = 0, \qquad 5\beta_1^4\beta_3 + 10\beta_1^3\beta_2 - \alpha^2\beta_1^3 + \beta_3 = 0$$

$$5\beta_1^4\beta_4 + 10\beta_1^2\beta_2(\beta_2^2 + 2\beta_1\beta_3) - 3\alpha^2\beta_1^2\beta_2 + \beta_4 = 0, \ldots$$

whose solutions are

$$\beta_1 = {}^1/_2\sqrt{2}\,(1 + i), \qquad \beta_2 = -{}^1/_4,$$

$$\beta_3 = {}^1/_8\sqrt{2}\,(1 - i)\,(\alpha^2 - {}^5/_8), \qquad \beta_4 = {}^1/_8 i\,({}^5/_4 - \alpha^2), \ldots$$

By substituting the found values of β_i into series (23) and then using Formulas (21) we find for the attenuation factor and the frequency of oscillation

$$\vartheta = -\left(\frac{1}{2}\pi^{\frac{1}{2}}\nu^{\frac{1}{2}}g^{\frac{1}{4}}h^{-\frac{3}{4}}\lambda^{-\frac{1}{2}} + \frac{\nu}{4h^2} + \frac{2\pi^2\nu}{\lambda^2} + \ldots\right)$$

$$\sigma = \frac{2\pi}{\lambda}\sqrt{gh} - \frac{1}{2}\pi^{\frac{1}{2}}\nu^{\frac{1}{2}}g^{\frac{1}{4}}h^{-\frac{3}{4}}\lambda^{-\frac{1}{2}} - \ldots \tag{24}$$

The particular case $\nu = 0$ is that of an ideal fluid. From the last formula it is evident that the frequency of oscillation in a viscous fluid is lower than that in an ideal one.

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